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Abstract

Solutions to the four-dimensional Euclidean Weyl equation in the background of a general JNR N-instanton are known to be normalizable and regular throughout 4-space. We show that these solutions are asymptotically given by a linear combination of simple singular solutions to the free Weyl equation, which can be interpreted as localized spinors. The 'spinorial' data parametrizing the asymptotics of the delocalized solutions to the Weyl equation in the presence of the instanton almost determine the background instanton, yet not completely. However, they capture the geometry and symmetry of the underlying instanton configuration.

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1. Introduction

It is a well-known fact that in the case of massless particles the Weyl equation, rather than the Dirac equation, provides an adequate description of particles with negative helicity, if we ignore positive helicity particles, or conversely. One finds that in two-dimensional Euclidean spacetime the Weyl equation is equivalent to the Cauchy–Riemann equations. Thus the class of solutions is given by the holomorphic functions. To interpret the solutions as localized classical particles, they must approach zero at infinity. If one also insists on their analyticity in the whole of Euclidean 2-space, the only possible solutions are the constants (Liouville's theorem). Hence, any *interesting* solutions that tend to zero at infinity must exhibit a singularity. In the simplest case, these solutions are therefore just given by $f(z) = a/(z - b)$, characterized by two complex constants, the residue a of the function and the position parameter b. However, the singularity makes the interpretation of these solutions as well-behaved particles again difficult.

In Euclidean 4-space we find a similar situation. Here the Weyl equation for negative helicity particles is equivalent to the so-called Cauchy–Riemann–Fueter equation, the quaternionic analogue of the Cauchy–Riemann equations. Just as holomorphic functions satisfy the latter, quaternionic *regular* functions satisfy the former. This definition of quaternionic regular functions is the basis of quaternionic analysis, which provides us with quaternionic analogues of, for example, Cauchy's theorem, Cauchy's integral formula and the Laurent expansion [1]. Since the proof of Liouville's theorem depends only on Cauchy's integral formula, there exists also the quaternionic analogue of this theorem. Hence, again any *interesting* regular function that approaches zero at infinity has to be singular in at least one point and thus, cannot easily be interpreted as a classical particle.

However, if one considers the Weyl equation in the background of an instanton, i.e. a topologically non-trivial solution to the four-dimensional Euclidean Yang–Mills equation, one finds that the instanton 'washes out' the singularity, leaving as a solution an everywhere nicely behaved particle. By this we mean that the asymptotic behaviour of the solution to the free Weyl equation and the solution in the instanton background agree to appropriate order, i.e. up to and including terms of $\mathcal{O}(r^{-4})$. Since the solution to the free equation and the solution in the instanton background are topologically inequivalent, a comparison of the two depends on a careful choice of gauge. It can be seen from the quaternionic Laurent series that the first term of the expansion, which is of order $\mathcal{O}(r^{-3})$, is analogous to a dipole field in three space dimensions, whereas the second is the analogue of a quadrupole field. The dipole term is characterized by one quaternionic constant, the quaternionic analogue of a residue. In contrast, the quadrupole term is characterized by three quaternionic constants, the interpretation of which is less clear. However, they can be shown to reflect the symmetry of the underlying instanton configuration.

In the following section, we will briefly explain the concept of an *instanton* [2], followed by the description of a special class of instantons, the so-called Jackiw–Nohl–Rebbi (JNR) instantons [3]. We will then describe the solutions to the four-dimensional Euclidean Weyl equation in the background field of an arbitrary JNR instanton, as investigated by Grossman [4]. We will then briefly summarize the theorems of quaternionic analysis [1] that are most important for us, before starting to describe our own results.

2. Euclidean Yang–Mills configurations

In four-dimensional Euclidean spacetime the Yang–Mills field equations possess localized solutions having a finite Euclidean action. These solutions are called *instantons*. Topologically distinct from the trivial absolute minimum of the action functional, instantons do not have a vanishing field strength $G_{\mu\nu}$. However, $G_{\mu\nu}$ has to vanish on the boundary of Euclidean 4-space, S^3_∞, which implies that the corresponding potential A_μ tends asymptotically to a pure gauge, thus defining a mapping from the 3-sphere at infinity into the manifold of the gauge group. Instantons are therefore characterized by a topological invariant N, the so-called Pontryagin index, which takes integer values.

In the case of an $SU(2)$ theory this index is the degree of the mapping of S^3_∞ into $SU(2) \cong S^3$, and is equal to the homotopy index, which characterizes the discrete infinity of homotopy classes of mappings $S^3 \to S^3$. Within each class N, the action is bounded from below by a constant multiple of $|N|$ and the absolute minimum value is attained when the field strength satisfies $G_{\mu\nu} = \pm * G_{\mu\nu}$, where $*G_{\mu\nu}$ is the dual field tensor given by $*G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}$.

Strictly speaking, *instantons* are self-dual solutions of the Yang–Mills equation whereas anti-self-dual solutions are referred to as *anti-instantons*. The fundamental difference between instantons and anti-instantons is the sign of the Pontryagin index, which is positive for instantons but negative for anti-instantons. For definiteness we will in the following only consider instantons though most results will be applicable to anti-instantons as well.

3. JNR instantons

The general N-instanton solution has been constructed implicitly by Atiyah *et al* in 1978 [5] for an arbitrary compact classical gauge group. In the case of $SU(2)$, $8N - 3$ parameters characterize the solutions. A subspace of these solutions can be constructed explicitly. These special instantons exhibit only $5N + 4$ parameters and were found by Jackiw, Nohl and Rebbi already in 1977 [3]. However, in the case of $N = 2$ this JNR solution to the self-duality equation is the most general one [6] since an $N = 2$ instanton is fully described by 13 physical parameters. The 14th parameter occurring in the JNR formula corresponds to a gauge transformation. An even more restrictive subspace consists of the 't Hooft instantons [7]. These are characterized by $5N$ parameters and thus give the most general $N = 1$ instanton. The $N = 1$ 't Hooft instanton is related to the JNR $N = 1$ instanton via a gauge transformation [8].

We begin by briefly summarizing the construction of the JNR N-instanton solution as described in [2, 3]. To simplify notation we consider the matrix representation of the gauge potential A_μ taking values in the Lie algebra of $SU(2)$,

$$A_\mu = A_\mu^a \frac{\sigma_a}{2\mathrm{i}}. \tag{3.1}$$

Here σ_a are the familiar Pauli matrices and A_μ^a, $a = 1, 2, 3$, are the components of the potential. The field strength is given by

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.2}$$

It is useful to define the following matrices:

$$\alpha_0 = 1_2 \qquad \alpha_i = -\mathrm{i}\sigma_{\mathrm{i}} \qquad \bar{\alpha}_0 = 1_2 \qquad \bar{\alpha}_i = \mathrm{i}\sigma_i \tag{3.3}$$

and

$$\bar{\sigma}_{\mu\nu} = \frac{1}{4\mathrm{i}}(\bar{\alpha}_\mu \alpha_\nu - \bar{\alpha}_\nu \alpha_\mu). \tag{3.4}$$

Note that $\bar{\sigma}_{\mu\nu}$ is both antisymmetric and anti-self-dual in its indices. In order to solve the self-duality equation for the field strength

$$G_{\mu\nu} = *G_{\mu\nu} \tag{3.5}$$

one makes an ansatz of the form

$$A_\mu = \mathrm{i}\bar{\sigma}_{\mu\nu} a_\nu \qquad \text{with} \quad a_\nu = \partial_\nu \ln \rho. \tag{3.6}$$

$\rho(\mathbf{x})$ is a scalar potential which must satisfy $\triangle\rho(\mathbf{x}) = 0$. Here $\triangle$ is the four-dimensional Euclidean Laplace operator. Note that the anti-self-dual symbols $\bar{\sigma}_{\mu\nu}$ in the potential A_μ lead to a self-dual field strength. One finds for $\rho(\mathbf{x})$ [1]

$$\rho(\mathbf{x}) = \sum_{k=1}^{N+1} \frac{\lambda_{(k)}^2}{(\mathbf{x} - \mathbf{x}_{(k)})^2}. \tag{3.7}$$

This solution when inserted in equation (3.6) yields an N-instanton solution with homotopy index N. Note that although ρ has a singularity at each point $\mathbf{x}_{(k)}$, the instanton field strength has no singularity.

It is important to note that the scalar potential ρ depends on $N + 1$ position parameters $\mathbf{x}_{(k)}$, and there is no simple relation between these and the positions of the N instantons. In contrast, a 't Hooft instanton has a scalar potential depending on N position parameters indeed

[1] 't Hooft's solution takes the form $\rho(\mathbf{x}) = 1 + \sum_{k=1}^{N} \frac{\lambda_{(k)}^2}{(\mathbf{x}-\mathbf{x}_{(k)})^2}$.

corresponding to the positions of the instantons. The other N JNR parameters, the weights $\lambda^2_{(k)}$, are indirectly related to the sizes of the instantons, but note that a common rescaling of the $\lambda^2_{(k)}$ does not affect the potential A_μ.

As shown by Jackiw, Nohl and Rebbi [3] the ansatz in equations (3.6) and (3.7) completely fixes the gauge, as long as the N points $\mathbf{x}_{(k)}$ do not lie on a circle (or on a straight line, which is a circle through the point at infinity). However, three points always lie on a circle and hence in the case of an $N = 2$ instanton one of the 14 parameters corresponds to a gauge transformation which moves the $\mathbf{x}_{(k)}$ around the circle.

4. Solutions to the massless Dirac equation in an *N*-instanton background

Solutions to the massless Dirac equation in the background of an N-instanton have been explicitly constructed by Grossman [4] with the gauge group being $SU(2)$. Later, Corrigan *et al* [9] and independently Osborn [10] derived these solutions for an $SU(n)$ or $Sp(k)$ gauge group, respectively. However, the latter solutions are only implicitly known. We will now briefly explain Grossman's construction, using, however, a slightly different notation.

The zero modes of the massless Dirac equation can be chosen to be chiral eigenfunctions of γ_5. According to Grossman it is possible to construct N solutions of negative helicity, when the field strength is self-dual[2]. These are all possible solutions because a vanishing theorem asserts that there are no solutions of positive helicity, and a version of the Atiyah–Singer index theorem states that the difference between the number of negative and positive helicity solutions must equal in absolute value the Pontryagin index N.

The massless Dirac equation in the background of an $SU(2)$ gauge potential A_μ is

$$(\partial_\mu + A_\mu)\gamma_\mu\psi = 0. \tag{4.1}$$

We choose a representation of the γ matrices in which $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is diagonal. This is given by

$$\gamma_0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & -\mathrm{i}\sigma_i \\ \mathrm{i}\sigma_i & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}. \tag{4.2}$$

Setting $\psi = \binom{\psi_R}{\psi_L}$ such that ψ_R and ψ_L are 2×2 matrices in spin and isospin, we get decoupled equations for the right-handed (negative helicity) and left-handed (positive helicity) spinors:

$$(\partial_\mu + A_\mu)\bar{\alpha}_\mu\psi_R = 0 \tag{4.3}$$

$$(\partial_\mu + A_\mu)\alpha_\mu\psi_L = 0. \tag{4.4}$$

It will be convenient to slightly rewrite equation (4.3). We have, preserving all matrix indices for the moment,

$$(\partial_\mu\delta_{ij} + (A_\mu)_{ij})(\bar{\alpha}_\mu)_{\beta\gamma}\psi_{\gamma j} = 0. \tag{4.5}$$

Using the fact that $\sigma_k^{\mathrm{t}} = \epsilon\sigma_k\epsilon$, where ϵ is the usual totally antisymmetric tensor with $\epsilon_{12} = 1$, and redefining the field $\psi'_R = \psi_R^{\mathrm{t}}\epsilon$, such that ψ'_R is a 2×2 matrix in isospin and spin, with components $\psi'^R_{j\alpha}$ (here $j = 1, 2$ is the isotopic index and $\alpha = 1, 2$ is the Lorentz index), equation (4.5) becomes

$$(\partial_\mu\delta_{ij} + (A_\mu)_{ij})\psi'_{j\beta}(\alpha_\mu)_{\beta\gamma} = 0 \tag{4.6}$$

[2] Grossman's actual derivation holds for positive helicity particles. However, as he points out, the analysis will likewise lead to N negative helicity solutions when considering self-dual fields $G_{\mu\nu}$, rather than anti-self-dual fields.

or in index-free notation

$$(\partial_\mu + A_\mu)\psi'_R \alpha_\mu = 0. \tag{4.7}$$

Note that all products occurring in equation (4.7) are now to be interpreted as matrix products. Defining

$$\phi^{(k)} = \frac{\lambda^2_{(k)}}{(\mathbf{x} - \mathbf{x}_{(k)})^2} \qquad k = 1, \ldots, N+1 \tag{4.8}$$

and

$$M^{(k)}_\mu = \rho^{1/2}\partial_\mu \left(\frac{\phi^{(k)}}{\rho}\right) \tag{4.9}$$

with ρ given by equation (3.7), one finds $N + 1$ solutions of the form

$$\psi'^{(k)}_R = M^{(k)}_\beta \bar{\alpha}_\beta. \tag{4.10}$$

However, since

$$\sum_{k=1}^{N+1} M^{(k)}_\mu = 0 \tag{4.11}$$

there are only N linearly independent solutions.

In the following we will briefly describe some major results of the theory of quaternionic regular functions, as they are solutions to the four-dimensional Weyl equation without a gauge field. We shall follow Sudbery's paper [1], which gives a self-contained and rigorous account of quaternionic analysis.

5. Quaternionic analysis

Quaternionic analysis was developed by Fueter and his collaborators in the years following 1935, when Fueter proposed the definition of *regular* for quaternionic functions [11].

A generic quaternion can be written as

$$q = 1x_0 + ix_1 + jx_2 + kx_3 \qquad \text{with} \quad x_\mu \in \mathrm{R} \quad \mu = 0, \ldots, 3 \tag{5.1}$$

where $1, i, j$ and k denote the elements of the standard basis for R^4. One defines the quaternionic product on R^4 as the R-bilinear product

$$\mathrm{R}^4 \times \mathrm{R}^4 \to \mathrm{R}^4 \qquad (q_1, q_2) \mapsto q_1 q_2 \tag{5.2}$$

with unit element 1, such that

$$i^2 = j^2 = k^2 = -1 \tag{5.3}$$

$$ij = k = -ji \qquad jk = i = -kj \qquad ki = j = -ik. \tag{5.4}$$

Here we will identify the subfield spanned by 1 with R. The linear space R^4 with the quaternionic product defines the real associative algebra H of the quaternions (see, e.g., [12]).

We will sometimes use $e_i, i = 1, 2, 3$, to denote the basic quaternions i, j and k, and e_0 to denote 1. Then equation (5.1) becomes

$$q = e_\mu x_\mu \tag{5.5}$$

where summation over repeated indices is implied.

Identifying the subfield spanned by 1 and k with the complex field C we will sometimes write

$$q = y + jz \tag{5.6}$$

where $y = x_0 + kx_3$ and $z = x_2 + kx_1$. The conjugate of q is $\bar{q} = x_0 - e_i x_i$ and the modulus is $|q| = \sqrt{q\bar{q}} = \sqrt{x_\mu x_\mu} \in \mathrm{R}$. We have $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$ and $q^{-1} = \bar{q}/|q|^2$.

The following definition of a *regular* function is the most convenient for our purposes (Sudbery gives a different definition, but shows that it is equivalent to this).

Definition 1 (the Cauchy–Riemann–Fueter equations). *A real-differentiable, quaternion-valued function $f : \mathrm{R}^4 \to \mathrm{H}$ is right-regular at q if and only if*

$$\frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k = 0. \tag{5.7}$$

It is left-regular at q if and only if

$$\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0. \tag{5.8}$$

The analysis of regular quaternionic functions is comparable to that of holomorphic functions, since the quaternionic analogues of Cauchy's theorem, Cauchy's integral theorem and the Laurent series exist for regular functions. As many of the standard theorems of complex analysis depend only on Cauchy's theorem, they also hold for quaternionic regular functions. An example of this is Liouville's theorem. Hence, a quaternionic function, which is regular in the whole of R^4 and that tends to zero as $|q| \to \infty$ must necessarily reduce to a constant. Also, regular functions automatically satisfy the four-dimensional Laplace equation. We will in the following consider only right-regular functions which we shall call simply *regular*[3].

Setting $q = y + jz$, as in equation (5.6), and $f = h + gj$, where g and h are complex-valued functions, we find that equation (5.7) is equivalent to the pair of complex equations

$$-\partial_{\bar{z}} g + \partial_{\bar{y}} h = 0 \qquad \partial_y g + \partial_z h = 0. \tag{5.9}$$

In the absence of a background field these complex equations correspond to equation (4.7), which is the Weyl equation for right-handed spinors. Note that it is permitted to multiply f by an arbitrary constant from the left.

For our purposes, Sudbery's discussion of regular power series is especially important. Sudbery introduces the following differential operator:

$$\partial_\nu = \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} = \frac{\partial^n}{\partial {x_1}^{n_1} \partial {x_2}^{n_2} \partial {x_3}^{n_3}}. \tag{5.10}$$

This calls for some explanation. ν is an unordered set of n integers $\{i_1, \ldots, i_n\}$ with $1 \leqslant i_r \leqslant 3$. The number of 1's in ν is given by n_1, the number of 2's by n_2 and the number of 3's by n_3 such that $n_1 + n_2 + n_3 = n$. Hence there are $\frac{1}{2}(n+1)(n+2)$ such sets ν and we will denote the collection of these sets as s_n. They are to be used as labels. If $n = 0$, i.e. $\nu = \emptyset$, we use the suffix 0 instead of $\emptyset$.

We will furthermore need the following two quaternion-valued functions:

$$G_\nu(q) = \partial_\nu G(q) \qquad \text{with} \quad G_0(q) \equiv G(q) = \frac{q^{-1}}{|q|^2} \tag{5.11}$$

[3] Sudbery chooses to develop the theory of quaternionic analysis for left-regular functions. However, his results are easily rewritten for right-regular functions.

and

$$P_\nu(q) = \frac{1}{n!} \sum \left(x_0 e_{i_1} - x_{i_1}\right) \cdots \left(x_0 e_{i_n} - x_{i_n}\right) \qquad \text{with} \quad P_0(q) = 1 \quad (5.12)$$

where the sum is over all $n!/(n_1!n_2!n_3!)$ different orderings of n_1 1's, n_2 2's and n_3 3's. Note that the P_ν are both right- and left-regular. We have

$$\frac{1}{2\pi^2} \int_S P_\mu(q) Dq G_\nu(q) = \delta_{\mu\nu} \quad (5.13)$$

and, since the P_ν are also left-regular, we have as well

$$\frac{1}{2\pi^2} \int_S G_\mu(q) Dq P_\nu(q) = \delta_{\mu\nu}. \quad (5.14)$$

Here S is any 3-sphere surrounding the origin and the measure Dq is the quaternion-valued 3-form

$$Dq = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \epsilon_{ijk} e_i \mathrm{d}x_0 \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k. \quad (5.15)$$

Sudbery then proves the following theorem (here slightly simplified):

Theorem 1 (the Laurent series)**.** *Suppose f is regular in an open set U except possibly at $q_0 \in U$. Then there is a neighbourhood N of q_0 such that if $q \in N$ and $q \neq q_0$, $f(q)$ can be represented by a series*

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in s_n} (a_\nu P_\nu(q - q_0) + b_\nu G_\nu(q - q_0)) \quad (5.16)$$

which converges uniformly in any hollow ball

$$\{q : r \leqslant |q - q_0| \leqslant R\} \qquad \textit{with} \quad r > 0 \quad \textit{which lies inside } N.$$

The coefficients a_ν and b_ν are given by

$$a_\nu = \frac{1}{2\pi^2} \int_C f(q) Dq G_\nu(q - q_0) \quad (5.17)$$

$$b_\nu = \frac{1}{2\pi^2} \int_C f(q) Dq P_\nu(q - q_0) \quad (5.18)$$

where C is any 3-sphere in U enclosing q_0.

6. The Dirac equation in quaternionic notation

As remarked in the previous section, the Weyl equation for right-handed spinors in the absence of an external field[4]

$$\partial_\mu \psi \alpha_\mu = 0 \quad (6.1)$$

corresponds to the complex version of the Cauchy–Riemann–Fueter equation, when one uses the complex variables $y = x_0 + \mathrm{i}x_3$, $z = x_2 + \mathrm{i}x_1$

$$-\partial_{\bar{z}} \psi_1 + \partial_{\bar{y}} \psi_2 = 0 \qquad \partial_y \psi_1 + \partial_z \psi_2 = 0.$$

Here (ψ_1, ψ_2) are the two complex components of ψ.

The correspondence of the Cauchy–Riemann–Fueter equation to the free Weyl equation can be seen more directly. Since the group of quaternions with absolute value 1, namely the

[4] We drop the prime and the subscript R from now on.

symplectic group $Sp(1)$, is isomorphic to $SU(2)$, we can rewrite equation (6.1) identifying $-\mathrm{i}\sigma_j$ with the quaternionic $e_j,\ j = 1, 2, 3$. This yields directly the Cauchy–Riemann–Fueter equation (5.7) in terms of quaternionic variables

$$\partial_\mu \psi_q e_\mu = 0 \tag{6.2}$$

where ψ_q is the single quaternionic function

$$\psi_q = \psi_2 + \psi_1 j. \tag{6.3}$$

We now turn to the Weyl equation (4.7) with a background field

$$(\partial_\mu + A_\mu)\psi\alpha_\mu = 0 \tag{6.4}$$

where ψ is again a 2×2 matrix in isospin and spin. Here, again, we can use the isomorphism $Sp(1) \cong SU(2)$ to rewrite equation (6.4). Applying this to both the Lorentz group and the gauge group will not be possible in general, since it would result in combining the four complex components of ψ into a single quaternionic function. However, for our purpose it will be convenient to impose the following $SU(2)$ gauge-invariant conditions on the four complex components of ψ, which will allow us to rewrite equation (6.4) in a purely quaternionic notation:

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ -\bar{\psi}_{12} & \bar{\psi}_{11} \end{pmatrix}. \tag{6.5}$$

Note that these conditions can be viewed as a Majorana condition. If we set

$$\Psi_M = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ -\bar{\psi}_{12} \\ \bar{\psi}_{11} \end{pmatrix} \tag{6.6}$$

we find

$$\Psi_M = \Psi_M^C \qquad \text{with} \quad \Psi_M^C \equiv C\bar{\Psi}_M^{\mathrm{t}}. \tag{6.7}$$

Here C is the charge conjugation operator given by $C = \gamma_0\gamma_2$ and $\bar{\Psi} = \Psi^\dagger\gamma_0$ is the usual Dirac conjugate spinor.

Setting $A_\mu^q = \frac{1}{2}e_a A_\mu^a$ we find that equation (6.4) simplifies to

$$\left(\partial_\mu + A_\mu^q\right)\psi_q e_\mu = 0 \tag{6.8}$$

where ψ_q is now a single quaternionic object, given by

$$\psi_q = \bar{\psi}_{11} - \bar{\psi}_{12} j = \psi_{22} + \psi_{21} j. \tag{6.9}$$

For the gauge transformation of ψ_q and A_μ^q we have

$$\psi_q \longrightarrow q\psi_q \tag{6.10}$$

$$A_\mu^q \longrightarrow qA_\mu^q q^{-1} - (\partial_\mu q)q^{-1} \tag{6.11}$$

with q a unit quaternion-valued function, i.e. $|q| = 1$.

As the simplest singular solution to (the complex) equation (6.1) we find

$$\psi = \frac{1}{(y\bar{y} + z\bar{z})^2}\begin{pmatrix} \bar{a}y + b\bar{z} & \bar{a}z - b\bar{y} \\ -a\bar{z} + \bar{b}y & a\bar{y} + \bar{b}z \end{pmatrix} \tag{6.12}$$

where we have already imposed the conditions (6.5). If we write $a = c_0 + \mathrm{i}c_3$, $b = c_2 + \mathrm{i}c_1$, where we assume c_μ to be the components of some real vector $\mathbf{c}$, we may write instead

$$\psi = \frac{(\mathbf{c}\cdot\boldsymbol{\alpha})(\mathbf{x}\cdot\bar{\boldsymbol{\alpha}})}{r^4} \tag{6.13}$$

with $r = \sqrt{\mathbf{x}^2}$ the distance from the origin.

In quaternionic notation we have (identifying as usual the quaternionic k with the complex i)

$$\psi_q = \frac{1}{(y\bar{y} + z\bar{z})^2}(a + jb)(\bar{y} - \bar{z}j). \tag{6.14}$$

Setting $q = y + jz$ as in equation (5.6) and defining $c_q = a + jb$ we have

$$\psi_q = c_q \frac{\bar{q}}{|q|^4} = c_q \frac{q^{-1}}{|q|^2} = c_q G(q) \tag{6.15}$$

with $G(q)$ as defined in equation (5.11). Note that this is the general 'dipole' term of the quaternionic Laurent expansion (5.16). Thus, ψ_q is regular except at the origin.

As we shall show in the following, the solutions to equation (6.4) in the presence of a JNR N-instanton can be written up to quadrupole order, i.e. up to and including terms of $\mathcal{O}(r^{-4})$, as a linear combination of shifted dipoles. However, this depends on a careful choice of gauge, as will be shown below.

7. $1/r$ expansion of Grossman's solution

In the background field of a JNR N-instanton there exist $N + 1$ (linearly dependent) solutions to the Weyl equation for right-handed spinors, which are given by equations (4.8)–(4.10)

$$\psi_{(k)} = M_\beta^{(k)} \bar{\alpha}_\beta \qquad k = 1, \ldots, N + 1.$$

Calculating these explicitly, we find for $M_\mu^{(k)}$

$$M_\mu^{(k)} = \frac{-2\lambda_{(k)}^2}{\rho^{3/2} \prod_{i=1}^{N+1} (\mathbf{x} - \mathbf{x}_{(i)})^4} \left[\sum_{i \neq k} \left(\prod_{j \neq i,k} (\mathbf{x} - \mathbf{x}_{(j)})^4 \right) \lambda_{(i)}^2 ((x_\mu - x_{(k)\mu})(\mathbf{x} - \mathbf{x}_{(i)})^2 \right.$$
$$\left. - (x_\mu - x_{(i)\mu})(\mathbf{x} - \mathbf{x}_{(k)})^2) \right] \tag{7.1}$$

with $\rho(\mathbf{x})$ given as before by equation (3.7).

In order to be able to compare this solution to the solution of the free Weyl equation discussed in the previous section, one has to investigate its asymptotic behaviour, i.e. its behaviour far away from the instanton. Thus we have to expand $M_\mu^{(k)}$ in powers of $1/r$. We shall write $M_\mu^{(k)}$ as the sum of a 'dipole' and a 'quadrupole' term, $m_{(k)\mu}^D$ and $m_{(k)\mu}^Q$, respectively, plus higher order contributions. It turns out that the first term of this expansion, the dipole term, is $\mathcal{O}(r^{-3})$, the quadrupole term $\mathcal{O}(r^{-4})$. Therefore

$$M_\mu^{(k)} = m_{(k)\mu}^D + m_{(k)\mu}^Q + \mathcal{O}\left(\frac{1}{r^5}\right). \tag{7.2}$$

Defining Λ as the sum of the weights $\lambda_{(i)}^2$ and $\mathbf{X}$ as the 'centre of mass' of the instanton

$$\Lambda = \sum_{i=1} \lambda_{(i)}^2 \qquad \text{and} \qquad \mathbf{X} = \frac{\sum_{i=1} \lambda_{(i)}^2 \mathbf{x}_{(i)}}{\Lambda} \tag{7.3}$$

the dipole contribution is given by

$$m_{(k)\mu}^D = -\frac{1}{r^3} \frac{2\lambda_{(k)}^2}{\Lambda^{1/2}} [(X_\mu - x_{(k)\mu}) - 2(\hat{\mathbf{x}} \cdot \mathbf{X} - \hat{\mathbf{x}} \cdot \mathbf{x}_{(k)})\hat{x}_\mu]. \tag{7.4}$$

If we define

$$\hat{n}^{(i)}_\mu = 2\hat{\mathbf{x}}\cdot\mathbf{x}_{(i)}\hat{x}_\mu - x_{(i)\mu} \qquad \text{and similarly} \quad \hat{n}^{(X)}_\mu = 2\hat{\mathbf{x}}\cdot\mathbf{X}\hat{x}_\mu - X_\mu \tag{7.5}$$

we can rewrite equation (7.4) as

$$m^D_{(k)\mu} = -\frac{1}{r^3}\frac{2\lambda^2_{(k)}}{\Lambda^{1/2}}\left[\hat{n}^{(k)}_\mu - \hat{n}^{(X)}_\mu\right]. \tag{7.6}$$

We find for the quadrupole term

$$\begin{aligned} m^Q_{(k)\mu} = -\frac{1}{r^4}\frac{2\lambda^2_{(k)}}{\Lambda^{1/2}}\Bigg[&\hat{x}_\mu\left(\frac{\sum_i \lambda^2_{(i)}\mathbf{x}^2_{(i)}}{\Lambda} - \mathbf{x}^2_{(k)}\right) + 2\left(\hat{\mathbf{x}}\cdot\mathbf{X}\hat{n}^{(k)}_\mu - \hat{\mathbf{x}}\cdot\mathbf{x}_{(k)}\hat{n}^{(X)}_\mu\right) \\ &+ 3\hat{\mathbf{x}}\cdot\mathbf{X}\left(\hat{n}^{(X)}_\mu - \hat{n}^{(k)}_\mu\right) + 4\left(\hat{\mathbf{x}}\cdot\mathbf{x}_{(k)}\hat{n}^{(k)}_\mu - \frac{\sum_i \lambda^2_{(i)}\hat{\mathbf{x}}\cdot\mathbf{x}_{(i)}\hat{n}^{(i)}_\mu}{\Lambda}\right)\Bigg]. \end{aligned} \tag{7.7}$$

If one now wants to compare this solution asymptotically to the solution of the Weyl equation in the absence of an instanton, one encounters the difficulty that the gauge potential A_μ tends asymptotically to a pure gauge rather than to the vacuum, which is topologically different.

It would be desirable to find a gauge in which A_μ approaches zero more rapidly than the pure gauge does. Because of the topologically different nature of the vacuum and the instanton, there exists no gauge transformation that is non-singular at all $\mathbf{x}$, relating the vacuum and the instanton. However, it turns out that one can find a *singular* gauge such that A_μ is (at most) $\mathcal{O}(r^{-3})$. An easy estimate then shows that in this singular gauge $\psi_{(k)}$ will be—up to and including quadrupole order—a solution not only to the Weyl equation in the instanton background but also to the free Weyl equation. Note that since the free Weyl equation is homogeneous, i.e. does not mix powers of r, the dipole term and the quadrupole term will be independently solutions to the free equation. The next task will obviously be to find the desired gauge transformation.

8. Singular gauge transformation

Under a gauge transformation, A_μ transforms as usual as

$$A_\mu \longrightarrow A'_\mu = UA_\mu U^{-1} - (\partial_\mu U)U^{-1}. \tag{8.1}$$

Expanding A_μ of equation (3.6) in powers of $1/r$ we find

$$A_\mu = -2\mathrm{i}\bar{\sigma}_{\mu\nu}\left[\frac{1}{r}\hat{x}_\nu + \frac{1}{r^2}\hat{n}^{(X)}_\nu\right] + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{8.2}$$

with $\hat{n}^{(X)}_\nu$ as in equation (7.5). If we define $U \in SU(2)$ as

$$U = \hat{x}_\nu\alpha_\nu \tag{8.3}$$

we have $U^{-1} = \hat{x}_\nu\bar{\alpha}_\nu$ and

$$U^{-1}\partial_\mu U = \frac{-2\mathrm{i}\bar{\sigma}_{\mu\nu}\hat{x}_\nu}{r}. \tag{8.4}$$

Thus gauge transforming A_μ with U cancels out the $1/r$ contribution to A_μ. The additional gauge transformation required to cancel out the $1/r^2$ contribution can be found as follows. We write $U_{\text{add}} = 1_2 + \epsilon$ for the additionally required $SU(2)$ gauge transformation, where we assume ϵ to be of $\mathcal{O}(r^{-1})$. The inverse is $U^{-1}_{\text{add}} = 1_2 - \epsilon$. Gauge transforming again we have

$$A'_\mu \longrightarrow A''_\mu = A'_\mu + \epsilon A'_\mu - A'_\mu \epsilon + \epsilon A'_\mu \epsilon - \partial_\mu \epsilon (1_2 - \epsilon). \tag{8.5}$$

Thus, we can read off that the $\mathcal{O}(r^{-2})$ contribution to A'_μ will be cancelled by $\partial_\mu \epsilon$. Hence, if we are able to write the $\mathcal{O}(r^{-2})$ contribution to A'_μ as the derivative of some function ϵ we will have obtained the required additional gauge transformation. Using the fact that

$$\delta_{\mu\nu} = \tfrac{1}{2}(\alpha_\mu \bar{\alpha}_\nu + \alpha_\nu \bar{\alpha}_\mu) \tag{8.6}$$

we find for the $\mathcal{O}(r^{-2})$ contribution to A'_μ

$$\begin{aligned} -2\mathrm{i} U \bar{\sigma}_{\mu\nu} U^{-1} \frac{\hat{n}_\nu^{(X)}}{r^2} &= \frac{1}{r^2}\left[-\hat{n}_\mu^{(X)} + \mathbf{X} \cdot \boldsymbol{\alpha}(2\hat{x}_\mu \hat{x}_\nu \bar{\alpha}_\nu - \bar{\alpha}_\mu)\right] \\ &= \partial_\mu \left[\frac{\hat{\mathbf{x}} \cdot \mathbf{X} - (\mathbf{X} \cdot \boldsymbol{\alpha})(\hat{\mathbf{x}} \cdot \bar{\boldsymbol{\alpha}})}{r}\right]. \end{aligned} \tag{8.7}$$

Thus the additional gauge transformation is given by

$$U_{\mathrm{add}} = 1_2 + \frac{\hat{\mathbf{x}} \cdot \mathbf{X} - (\mathbf{X} \cdot \boldsymbol{\alpha})(\hat{\mathbf{x}} \cdot \bar{\boldsymbol{\alpha}})}{r}. \tag{8.8}$$

Defining the self-dual symbols $\sigma_{\mu\nu}$ analogously to the anti-self-dual symbols $\bar{\sigma}_{\mu\nu}$

$$\sigma_{\mu\nu} = \frac{1}{4\mathrm{i}}(\alpha_\mu \bar{\alpha}_\nu - \alpha_\nu \bar{\alpha}_\mu) \tag{8.9}$$

it is easy to see that U_{add} is indeed an $SU(2)$ gauge transformation, since we may write

$$\hat{\mathbf{x}} \cdot \mathbf{X} - (\mathbf{X} \cdot \boldsymbol{\alpha})(\hat{\mathbf{x}} \cdot \bar{\boldsymbol{\alpha}}) = -2\mathrm{i}\sigma_{\mu\nu} X_\mu \hat{x}_\nu \tag{8.10}$$

where we have used

$$\alpha_\mu \bar{\alpha}_\nu = 2\mathrm{i}\sigma_{\mu\nu} + \delta_{\mu\nu}. \tag{8.11}$$

Note that the $\sigma_{\mu\nu}$ take their values in the Lie algebra of $SU(2)$ as do the $\bar{\sigma}_{\mu\nu}$.

It is remarkable that the additional gauge transformation depends only on the centre of mass $\mathbf{X}$ of the instanton. In the centre of mass frame, where $\mathbf{X} = \mathbf{0}$, the additional gauge transformation is the identity. Thus in this frame the quadrupole term of the spinor is a solution to the free equation even without an additional gauge transformation.

The basic quantity $\hat{n}_\mu^{(k)} \bar{\alpha}_\mu$, defined in equation (7.5), gauge transformed by U, is given by

$$\begin{aligned} \hat{\mathcal{N}}_s^{(k)} &\equiv \hat{x}_\mu \alpha_\mu \hat{n}_\nu^{(k)} \bar{\alpha}_\nu \\ &= 2\hat{\mathbf{x}} \cdot \mathbf{x}_{(k)} - (\hat{\mathbf{x}} \cdot \boldsymbol{\alpha})(\mathbf{x}_{(k)} \cdot \bar{\boldsymbol{\alpha}}) \\ &= (\mathbf{x}_{(k)} \cdot \boldsymbol{\alpha})(\hat{\mathbf{x}} \cdot \bar{\boldsymbol{\alpha}}). \end{aligned} \tag{8.12}$$

Comparing this to the singular solution discussed previously (equation (6.13)) we find that $\hat{\mathcal{N}}_s^{(k)}$ is precisely the matrix occurring there, with the components of the vector $\mathbf{c}$ determined by the components of the position parameter $\mathbf{x}_{(k)}$ of the instanton.

We write $\tilde{\psi}_{(k)}$ for $\psi_{(k)}$ fully gauge transformed by U and U_{add}. Then the dipole contribution $\psi_{(k)}^D$ to $\tilde{\psi}_{(k)}$ is

$$\psi_{(k)}^D = \hat{\mathbf{x}} \cdot \boldsymbol{\alpha} m_{(k)\mu}^D \bar{\alpha}_\mu = -\frac{1}{r^3} \frac{2\lambda_{(k)}^2}{\Lambda^{1/2}} \left[\hat{\mathcal{N}}_s^{(k)} - \hat{\mathcal{N}}_s^{(X)}\right]. \tag{8.13}$$

Note that the dipole contribution $\psi_{(k)}^D$ will be unaffected by the additional gauge transformation U_{add}. However, $\psi_{(k)}^D$ yields a contribution to the quadrupole term, when one performs the additional gauge transformation $U_{\rm add}$. The quadrupole contribution $\psi^Q_{(k)}$ to $\tilde{\psi}_{(k)}$ is given by

$$\begin{aligned}\psi^Q_{(k)} &= \hat{\mathbf{x}}\cdot\alpha m^Q_{(k)\mu}\bar{\alpha}_\mu + \frac{\hat{\mathbf{x}}\cdot\mathbf{X} - (\mathbf{X}\cdot\alpha)(\hat{\mathbf{x}}\cdot\bar{\alpha})}{r}\psi^D_{(k)} \\ &= -\frac{1}{r^4}\frac{2\lambda^2_{(k)}}{\Lambda^{1/2}}\left[4\left(\hat{\mathbf{x}}\cdot\mathbf{X}\hat{\mathcal{N}}^{(X)}_s - \hat{\mathbf{x}}\cdot\mathbf{x}_{(k)}\hat{\mathcal{N}}^{(X)}_s + \hat{\mathbf{x}}\cdot\mathbf{x}_{(k)}\hat{\mathcal{N}}^{(k)}_s - \frac{1}{\Lambda}\sum_i \lambda^2_{(i)}\hat{\mathbf{x}}\cdot\mathbf{x}_{(i)}\hat{\mathcal{N}}^{(i)}_s\right)\right. \\ &\quad\left. + \left(-\mathbf{X}^2 + (\mathbf{X}\cdot\alpha)(\mathbf{x}_{(k)}\cdot\bar{\alpha}) - \mathbf{x}^2_{(k)} + \frac{\sum_i \lambda^2_{(i)}\mathbf{x}^2_{(i)}}{\Lambda}\right)\right].\end{aligned} \tag{8.14}$$

To derive this we have used

$$\mathbf{X}\cdot\alpha\hat{n}^{(k)}_\mu\bar{\alpha}_\mu = 2\hat{\mathbf{x}}\cdot\mathbf{x}_{(k)}\hat{\mathcal{N}}^{(X)}_s - (\mathbf{X}\cdot\alpha)(\mathbf{x}_{(k)}\cdot\bar{\alpha}) \tag{8.15}$$

and

$$(\mathbf{x}\cdot\alpha)(\mathbf{x}\cdot\bar{\alpha}) = \mathbf{x}^2. \tag{8.16}$$

We have in total

$$\tilde{\psi}_{(k)} = \psi^D_{(k)} + \psi^Q_{(k)} + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{8.17}$$

where $\psi^D_{(k)} + \psi^Q_{(k)}$ should be a solution not only to the Dirac equation in the instanton background but also to the free equation. That this is indeed the case may be verified by considering solution (6.13) with its pole shifted by $\mathbf{x}_{(j)}$ and with $\mathbf{c} = \mathbf{x}_{(i)}$, which is obviously still a solution to the free equation. This shifted solution has the expansion

$$\begin{aligned}&\frac{[(\mathbf{x}_{(i)}\cdot\alpha)(\mathbf{x}\cdot\bar{\alpha}) - (\mathbf{x}_{(i)}\cdot\alpha)(\mathbf{x}_{(j)}\cdot\bar{\alpha})]}{(\mathbf{x}-\mathbf{x}_{(j)})^4} \\ &\quad= \frac{1}{r^3}\hat{\mathcal{N}}^{(i)}_s + \frac{1}{r^4}\left[4\hat{\mathbf{x}}\cdot\mathbf{x}_{(j)}\hat{\mathcal{N}}^{(i)}_s - (\mathbf{x}_{(i)}\cdot\alpha)(\mathbf{x}_{(j)}\cdot\bar{\alpha})\right] + \mathcal{O}\left(\frac{1}{r^5}\right).\end{aligned} \tag{8.18}$$

One now easily verifies that both the dipole term $\psi^D_{(k)}$ and the quadrupole term $\psi^Q_{(k)}$ separately solve the free equation, since one finds $\psi^D_{(k)}$ to be a linear combination of terms of the form $r^{-3}\hat{\mathcal{N}}^{(i)}_s$ and $\psi^Q_{(k)}$ to be a linear combination of terms of the form $r^{-4}\big[4\hat{\mathbf{x}}\cdot\mathbf{x}_{(j)}\hat{\mathcal{N}}^{(i)}_s - (\mathbf{x}_{(i)}\cdot\alpha)(\mathbf{x}_{(j)}\cdot\bar{\alpha})\big]$.

Thus we find that the solution to the Dirac equation in the instanton background in this singular gauge coincides to appropriate order with a linear combination of singular solutions to the free equation as given by equation (8.18). In this sense the instanton 'washes out' the singularity of this latter solution yielding an everywhere nicely behaved particle.

In the last section we will compare the quaternionic version of equation (8.18), given by

$$\frac{q_{(i)}(\bar{q} - \bar{q}_{(j)})}{|q - q_{(j)}|^4} = q_{(i)}\left[\frac{\hat{\bar{q}}}{|q|^3} + \frac{4\hat{\mathbf{x}}\cdot\mathbf{x}_{(j)}\hat{\bar{q}} - \bar{q}_{(j)}}{|q|^4}\right] + \mathcal{O}\left(\frac{1}{|q|^5}\right) \tag{8.19}$$

where $q_{(i)} = x_{(i)\mu}e_\mu$ and similarly $\hat{q} = \hat{x}_\mu e_\mu$, as well as the asymptotic fields $\tilde{\psi}_{(k)}$ in quaternionic notation, to the general quaternionic Laurent expansion. Investigating some special $N = 2$ instantons we will be able to show that the constants b_μ occurring in the quaternionic Laurent series reflect the symmetry of the underlying instanton configuration. First we will however briefly discuss the case of an $N = 1$ instanton.

9. N = 1 instanton in singular gauge

In the case of an $N = 1$ instanton, there exist two linearly dependent solutions to the Weyl equation, the sum of which is equal to zero. Thus the two solutions differ only in sign. However, the JNR formula for an $N = 1$ instanton is largely redundant, whereas the 't Hooft formula yields already the most general $N = 1$ instanton. As stated previously a JNR $N = 1$ instanton is related to a 't Hooft $N = 1$ instanton via a gauge transformation. Explicitly (see [8]), consider a JNR instanton with potential

$$\rho = \frac{\lambda_{(1)}^2}{(\mathbf{x} - \mathbf{x}_{(1)})^2} + \frac{\lambda_{(2)}^2}{(\mathbf{x} - \mathbf{x}_{(2)})^2} \tag{9.1}$$

and a 't Hooft instanton with potential

$$\rho_{\text{'t Hooft}} = 1 + \frac{\lambda_{(0)}^2}{(\mathbf{x} - \mathbf{x}_{(0)})^2} \tag{9.2}$$

where

$$\mathbf{x}_{(0)} = \frac{\lambda_{(1)}^2 \mathbf{x}_{(2)} + \lambda_{(2)}^2 \mathbf{x}_{(1)}}{\Lambda} \qquad \text{and} \qquad \lambda_{(0)}^2 = \frac{\lambda_{(1)}^2 \lambda_{(2)}^2}{\Lambda^2} |\mathbf{x}_{(2)} - \mathbf{x}_{(1)}|^2. \tag{9.3}$$

The JNR $N = 1$ instanton may then be obtained from the 't Hooft $N = 1$ instanton by conjugating with the $SU(2)$ matrix

$$U_0 = \frac{x_{(2)\mu} - x_{(1)\mu}}{|\mathbf{x}_{(2)} - \mathbf{x}_{(1)}|} \alpha_\mu \tag{9.4}$$

or vice versa by conjugating with the inverse of U_0. This can be verified by calculating the gauge transform of the asymptotic field $\tilde{\psi}_{(k)}$, $k = 1, 2$, given by equation (8.17), with the gauge transformation given by U_0^{-1}. Thus one finds the solution to the Weyl equation in the background of the JNR $N = 1$ instanton to be gauge equivalent to the solution to the Weyl equation in the background of the 't Hooft $N = 1$ instanton. The latter solution is given by

$$\psi_{\text{'t Hooft}} = \pm 2\lambda_{(0)}^2 \frac{(x_\mu - x_{(0)\mu})\bar{\alpha}_\mu}{|\mathbf{x} - \mathbf{x}_{(0)}| \left((\mathbf{x} - \mathbf{x}_{(0)})^2 + \lambda_{(0)}^2\right)^{3/2}}. \tag{9.5}$$

However, it is well known [3] that the JNR $N = 1$ instanton and the 't Hooft $N = 1$ instanton are related not only by a gauge transformation but also by a limiting process. Taking the JNR scalar potential

$$\rho(\mathbf{x}) = \sum_{i=1}^{N+1} \frac{\lambda_{(i)}^2}{(\mathbf{x} - \mathbf{x}_{(i)})^2}$$

the 't Hooft scalar potential

$$\rho(\mathbf{x}) = 1 + \sum_{i=1}^{N} \frac{\lambda_{(i)}^2}{(\mathbf{x} - \mathbf{x}_{(i)})^2}$$

may be regained in the limit $|\mathbf{x}_{(N+1)}| \to \infty$, $\lambda_{(N+1)}^2 \to \infty$ with $\lambda_{(N+1)}^2 / |\mathbf{x}_{(N+1)}| \to 1$. Similarly, one may obtain 't Hooft's solution to the Weyl equation in the presence of an $N = 1$ instanton, equation (9.5), via the same limiting process from Grossman's solution equation (4.10) to the Weyl equation in the background of a JNR $N = 1$ instanton. Thus the formula

$$\psi_{\text{'t Hooft}}^{(k)} = \rho_{\text{'t Hooft}}^{1/2} \partial_\mu \left(\frac{\phi_{\text{'t Hooft}}^{(k)}}{\rho_{\text{'t Hooft}}} \right) \bar{\alpha}_\mu \qquad k = 1, 2 \tag{9.6}$$

with

$$\phi^{(1)}_{\text{'t Hooft}} = \frac{\lambda^2_{(0)}}{(\mathbf{x} - \mathbf{x}_{(0)})^2} \qquad \phi^{(2)}_{\text{'t Hooft}} = 1 \tag{9.7}$$

and with $\rho_{\text{'t Hooft}}$ given by equation (9.2), will yield precisely 't Hooft's solution as stated in equation (9.5). Expanding equation (9.5) we have

$$\psi_{\text{'t Hooft}} = \pm 2\lambda^2_{(0)} \left(\frac{\hat{x}_\mu \bar{\alpha}_\mu}{r^3} + \frac{4\hat{\mathbf{x}} \cdot \mathbf{x}_{(0)} \hat{x}_\mu \bar{\alpha}_\mu - x_{(0)\mu} \bar{\alpha}_\mu}{r^4} \right) + \mathcal{O}\left(\frac{1}{r^5}\right). \tag{9.8}$$

Comparing this to equation (8.18), one finds that the expansion of 't Hooft's solution agrees with the expansion of the singular solution with shifted pole apart from a constant gauge transformation. Note that if one includes this rigid gauge transformation in the solution, the 't Hooft $N = 1$ instanton is characterized by eight rather than by five parameters. By comparison, if we ignore an overall multiplicative real constant, the number of parameters in equation (8.18) is seven in total, when the additional rigid gauge transformation is included. Thus the dipole and quadrupole terms together almost determine the background $N = 1$ instanton but not completely.

It will be generally convenient to include the three parameters arising from rigid gauge transformations in the number of parameters characterizing an instanton, though they are not of physical significance. One then finds that the general ADHM N-instanton is characterized by $8N$ parameters, the JNR N-instanton by $5N + 7$ and the 't Hooft N-instanton by $5N + 3$.

10. $N = 2$ instantons and interpretation of constants

We shall now investigate the dipole and quadrupole terms of the quaternionic Laurent expansion a bit further. As stated previously, the dipole term of this series, given by equation (6.15), is characterized by one constant c_q which is the quaternionic analogue of a residue. However, the next term of the expansion, the quadrupole term, is instead characterized by three constants, the interpretation of which is less clear. The quadrupole term of the Laurent series is given by

$$Q \equiv \sum_{i=1}^{3} b_i \partial_i G(q) \tag{10.1}$$

with the b_i given by equation (5.18). Calculating the derivative of $G(q)$ with respect to x_i explicitly we find

$$\frac{\partial}{\partial x_i} G(q) = -\left[\frac{e_i}{|q|^4} + \frac{4\hat{\bar{q}}}{|q|^4} \hat{x}_i \right]. \tag{10.2}$$

Note that only derivatives with respect to the spatial coordinates x_i occur here, since the derivative of $G(q)$ with respect to the time coordinate x_0 is determined via the Cauchy–Riemann–Fueter equation, once the spatial derivatives are known. However, this explicitly breaks the symmetry between the time coordinate and the spatial coordinates. Setting

$$\tilde{b}_0 e_i - \tilde{b}_i \equiv b_i \qquad \text{with} \quad \tilde{b}_\mu \in \mathrm{H} \tag{10.3}$$

we may restore this symmetry in the equations. We then find for the quadrupole term

$$\begin{aligned} Q &= \sum_{i=1}^{3} -b_i \left[\frac{e_i}{|q|^4} + \frac{4\hat{\bar{q}}}{|q|^4} \hat{x}_i \right] \\ &= \frac{-\left[\tilde{b}_0 - \sum_{i=1}^{3} \tilde{b}_i e_i\right]}{|q|^4} + \frac{4\left[\sum_{\mu=0}^{3} \hat{x}_\mu \tilde{b}_\mu\right]\hat{\bar{q}}}{|q|^4} \end{aligned} \tag{10.4}$$

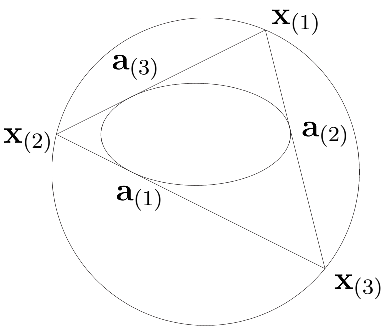

Figure 1. The circle and ellipse associated with an $N = 2$ instanton in R^4 and one member of the porism of triangles with vertices $\mathbf{x}_{(1)}$, $\mathbf{x}_{(2)}$ and $\mathbf{x}_{(3)}$ on the circle and tangent to the ellipse at $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$ and $\mathbf{a}_{(3)}$.

where we have used $\sum_{i=1}^{3} e_i \hat{x}_i \,\hat{\bar{q}} = 1 - \hat{x}_0 \hat{\bar{q}}$ to derive the last line. We may simplify notation further assuming the $\tilde{b}_\mu$ to be the components of some quaternionic-valued vector $\tilde{\mathbf{b}}$ and writing $\tilde{b}_q = \tilde{b}_0 + \sum_{i=1}^{3} \tilde{b}_i e_i$ and $\bar{\tilde{b}}_q = \tilde{b}_0 - \sum_{i=1}^{3} \tilde{b}_i e_i$, respectively. We then have

$$Q = \frac{-\bar{\tilde{b}}_q}{|q|^4} + 4\hat{\mathbf{x}} \cdot \tilde{\mathbf{b}} \frac{\hat{\bar{q}}}{|q|^4}. \tag{10.5}$$

However, one should note that this notation is somewhat misleading, since the components $\tilde{b}_\mu$ of $\tilde{b}_q$ are themselves quaternions. Therefore $\bar{\tilde{b}}_q$ will in general not be the quaternionic conjugate of $\tilde{b}_q$. Nevertheless this notation proves to be useful, as can be seen directly when comparing equation (10.5) to the $\mathcal{O}(r^{-4})$ contribution to equation (8.19). These two expressions have a functionally similar form, which will allow us to read off the constants $\tilde{b}_\mu$ for the quadrupole term of some specific instanton configuration.

Note that the number of parameters occurring in the Laurent series up to quadrupole order is 15, if we ignore an overall multiplicative real constant, but include rigid gauge transformations. This should be compared with 16 parameters describing an $N = 2$ instanton if one includes rigid gauge transformations. Hence, as in the $N = 1$ instanton case, the dipole and quadrupole terms together almost determine the background instanton but not completely. In the remainder we will show that the parameters $\tilde{b}_\mu$ are related to the symmetry of the underlying two-instanton configuration.

With each $N = 2$ instanton in R^4 there is an associated pair of a circle and an ellipse, which satisfies the Poncelet condition, which means that there is a porism (or one-parameter family) of triangles with vertices on the circle and tangent to the ellipse, see figure 1. In fact, if one such triangle exists then there is automatically a porism of triangles and any point on the circle may be a vertex. From this geometrical data one may reconstruct the instanton fields in JNR form (see [8]) by choosing one triangle of the porism with vertices $\mathbf{x}_{(1)}$, $\mathbf{x}_{(2)}$ and $\mathbf{x}_{(3)}$ on the circle, and tangent to the ellipse at $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$ and $\mathbf{a}_{(3)}$. Then define weights $\lambda_{(1)}^2$, $\lambda_{(2)}^2$ and $\lambda_{(3)}^2$ up to a common multiple, by

$$\frac{\lambda_{(1)}^2}{\lambda_{(2)}^2} = \frac{|\mathbf{x}_{(1)} - \mathbf{a}_{(3)}|}{|\mathbf{a}_{(3)} - \mathbf{x}_{(2)}|} \quad \text{etc.} \tag{10.6}$$

The JNR potential is then given by

$$\rho = \sum_{i=1}^{3} \frac{\lambda_{(i)}^2}{(\mathbf{x} - \mathbf{x}_{(i)})^2}. \tag{10.7}$$

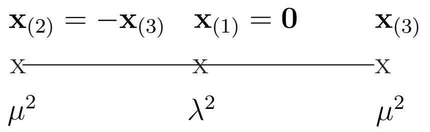

Figure 2. The three vertices associated with an $N = 2$ instanton in R^4 are chosen to be collinear. This is the limiting case of a general $N = 2$ instanton in R^4, for which the three vertices lie on a circle.

If another triangle of the porism is chosen, then a different expression for the scalar potential ρ is obtained, but the instanton will change only by a gauge transformation. This corresponds to the gauge invariance noted by Jackiw, Nohl and Rebbi, which moving the vertices around the circle will change the instanton only by a gauge transformation.

As the first example we will consider the limiting case of an $N = 2$ instanton, where $\mathbf{x}_{(1)}, \mathbf{x}_{(2)}$ and $\mathbf{x}_{(3)}$ are collinear, with $\mathbf{x}_{(1)} = \mathbf{0}$ and $\mathbf{x}_{(2)} = -\mathbf{x}_{(3)}$, as shown in figure 2. Note that in this limiting case the associated ellipse becomes degenerate. The scale parameters $\lambda^2_{(i)}$ are given by $\lambda^2_{(1)} = \lambda^2$ and $\lambda^2_{(2)} = \lambda^2_{(3)} = \mu^2$. Thus the centre of mass is $\mathbf{X} = \mathbf{0}$, and the sum of the weights is given by $\Lambda = \lambda^2 + 2\mu^2$.

There are three solutions $\tilde{\psi}_{(k)}$, whose dipole and quadrupole terms are given by equations (8.13) and (8.14), respectively. However, since $\tilde{\psi}_{(1)} + \tilde{\psi}_{(2)} + \tilde{\psi}_{(3)} = 0$, these three solutions are linearly dependent. Thus it is convenient to consider suitable linear combinations, which we will choose to reflect the symmetry of the instanton. These are

$$\tilde{\psi}_{(a)} = \tilde{\psi}_{(2)} - \tilde{\psi}_{(3)} \qquad \text{and} \qquad \tilde{\psi}_{(b)} = \tilde{\psi}_{(1)}. \tag{10.8}$$

We then find for the expansion of $\tilde{\psi}_{(a)}$

$$\tilde{\psi}_{(a)} = -\frac{4\mu^2}{\Lambda^{1/2}} \frac{1}{r^3} \hat{\mathcal{N}}_s^{(2)} + \mathcal{O}\left(\frac{1}{r^5}\right). \tag{10.9}$$

Here we have used that since $\hat{\mathcal{N}}_s^{(k)} = (\mathbf{x}_{(k)} \cdot \boldsymbol{\alpha})(\hat{\mathbf{x}} \cdot \bar{\boldsymbol{\alpha}})$ and $\mathbf{x}_{(3)} = -\mathbf{x}_{(2)}$, therefore $\hat{\mathcal{N}}_s^{(3)} = -\hat{\mathcal{N}}_s^{(2)}$. Note that the quadrupole contribution to $\tilde{\psi}_{(a)}$ vanishes. For $\tilde{\psi}_{(b)}$ we find instead

$$\tilde{\psi}_{(b)} = -\frac{1}{r^4} \frac{2\lambda^2\mu^2}{\Lambda^{3/2}} \left[-8\hat{\mathbf{x}} \cdot \mathbf{x}_{(2)} \hat{\mathcal{N}}_s^{(2)} + 2\mathbf{x}^2_{(2)}\right] + \mathcal{O}\left(\frac{1}{r^5}\right). \tag{10.10}$$

Here the dipole contribution is equal to zero. Note that $\tilde{\psi}_{(b)}$ can be written as the sum of two shifted dipoles, namely

$$\tilde{\psi}_{(b)} \approx -\frac{2\lambda^2\mu^2}{\Lambda^{3/2}} \left[\frac{(\mathbf{x}_{(2)} \cdot \boldsymbol{\alpha})(\mathbf{x} \cdot \bar{\boldsymbol{\alpha}} - \mathbf{x}_{(2)} \cdot \bar{\boldsymbol{\alpha}})}{(\mathbf{x} - \mathbf{x}_{(2)})^4} + \frac{(\mathbf{x}_{(3)} \cdot \boldsymbol{\alpha})(\mathbf{x} \cdot \bar{\boldsymbol{\alpha}} - \mathbf{x}_{(3)} \cdot \bar{\boldsymbol{\alpha}})}{(\mathbf{x} - \mathbf{x}_{(3)})^4}\right]. \tag{10.11}$$

This will give the correct expansion of $\tilde{\psi}_{(b)}$ at least up to quadrupole order. In contrast, $\tilde{\psi}_{(a)}$ can be interpreted as a pure dipole at $\mathbf{x}_{(1)} = \mathbf{0}$.

We have for $\tilde{\psi}_{(a)}, \tilde{\psi}_{(b)}$, respectively, in quaternionic notation

$$\tilde{\psi}_{(a)} = -\frac{4\mu^2}{\Lambda^{1/2}} q_{(2)} \frac{\hat{\bar{q}}}{|q|^3} + \mathcal{O}\left(\frac{1}{|q|^5}\right) \tag{10.12}$$

$$\tilde{\psi}_{(b)} = -\frac{4\lambda^2\mu^2}{\Lambda^{3/2}} q_{(2)} \frac{[-4\hat{\mathbf{x}} \cdot \mathbf{x}_{(2)} \hat{\bar{q}} + \bar{q}_{(2)}]}{|q|^4} + \mathcal{O}\left(\frac{1}{|q|^5}\right). \tag{10.13}$$

Now we are able to read off the constants $\tilde{b}_\mu$ characterizing the quadrupole term of the quaternionic Laurent series. For $\tilde{\psi}_{(b)}$,

$$\tilde{b}_\mu = \frac{4\lambda^2\mu^2}{\Lambda^{3/2}} q_{(2)} x_{(2)\mu} = \frac{4\lambda^2\mu^2}{\Lambda^{3/2}} x_{(2)\mu} x_{(2)\nu} e_\nu. \tag{10.14}$$

One easily checks that this gives also the correct constant contribution to the quadrupole term. For $\tilde{\psi}_{(a)}$ the $\tilde{b}_\mu$ are all zero as there is no quadrupole term. The constant $q_{(2)}$ can be interpreted as the residue of $\tilde{\psi}_{(a)}$. Note that, if one chooses the vertices to lie on some particular axis, say the x_μ-axis, only one of the parameters is not equal to zero, namely $\tilde{b}_\mu$, which is proportional to e_μ.

This nicely reflects the symmetry of the configuration of vertices, if one assumes a generic quaternion $q = x_0 + x_i e_i$ to be invariant under the following (generalized) rotations of R^3. Consider, for example, a rotation of R^3, which means $x_i \longrightarrow R_{ij} x_j$ with $R \in SO(3)$. Note that the quaternionic basis vectors e_i are in a sense arbitrary, since one may take linear combinations of them leading to an equivalent description of H. If we identify the space spanned by e_i with the space spanned by $R^{\mathrm{t}}_{ji} e_i$, the map of R^3 into the space of the pure quaternions with $x_0 = 0$ is spherically symmetric under the orthogonal transformation R, since then $q = x_i e_i \mapsto q = R_{ij} x_j e_i = x_j R^{\mathrm{t}}_{ji} e_i = x_j \tilde{e}_j \equiv x_j e_j$. In other words, under such transformations the generic quaternion q will be invariant.

For example, the general dipole term of the Laurent expansion L_D will under a rotation $x_i \mapsto R_{ij} x_j$ transform as

$$L_D = (c_0 + c_i e_i)\frac{\bar{q}}{|q|^4} \mapsto (c_0 + c_i R_{ji} \tilde{e}_j)\frac{x_0 - x_i \tilde{e}_i}{|q|^4} \equiv (c_0 + c_i R_{ji} e_j)\frac{\hat{\bar{q}}}{|q|^3}. \tag{10.15}$$

If we write the product of two basic quaternions

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \tag{10.16}$$

in terms of the new basis, we find (since $\det(R) = 1$)

$$e_i e_j = R_{mi} \tilde{e}_m R_{nj} \tilde{e}_n = -\delta_{ij} + \det(R)\epsilon_{ijn} R_{mn} \tilde{e}_m = -\delta_{ij} + \epsilon_{ijn} e_n. \tag{10.17}$$

We are now in a position to investigate the symmetry properties under rotations of solutions $\tilde{\psi}_{(a)}$ and $\tilde{\psi}_{(b)}$, equations (10.12) and (10.13), respectively. If one chooses, for example, the vertices in the above example to lie on the x_0-axis, the background instanton is spherically symmetric in the spatial directions. But under a spatial rotation in the above sense, $\tilde{\psi}_{(a)}$ remains invariant as does $\tilde{\psi}_{(b)}$. Thus $\tilde{\psi}_{(a)}$ and $\tilde{\psi}_{(b)}$ show the same symmetry properties as the background.

As our next example we will consider an instanton, where $\mathbf{x}_{(1)}$, $\mathbf{x}_{(2)}$ and $\mathbf{x}_{(3)}$ lie at the vertices of an equilateral triangle, such that $\sum \mathbf{x}_{(i)} = \mathbf{0}$, $\mathbf{x}^2_{(i)} = 1$ and where the weights are all equal, $\lambda^2_{(i)} = \lambda^2$. For definiteness we shall assume that the triangle lies in the (x_1, x_2) plane, see figure 3. We have

$$\mathbf{x}_{(1)} = \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \qquad \mathbf{x}_{(2)} = \begin{pmatrix} 0 \\ -\frac{1}{2}(\cos\alpha + \sqrt{3}\sin\alpha) \\ -\frac{1}{2}(\sin\alpha - \sqrt{3}\cos\alpha) \\ 0 \end{pmatrix} \qquad \mathbf{x}_{(3)} = -(\mathbf{x}_{(1)} + \mathbf{x}_{(2)}). \tag{10.18}$$

Here we have explicitly accounted for the fact that moving the vertices around the circle corresponds to a gauge transformation. Thus all configurations arising from different values of α in equation (10.18) are related via gauge transformations. Here again we shall consider suitable linear combinations of the three linearly dependent solutions $\tilde{\psi}_{(k)}$, respectively,

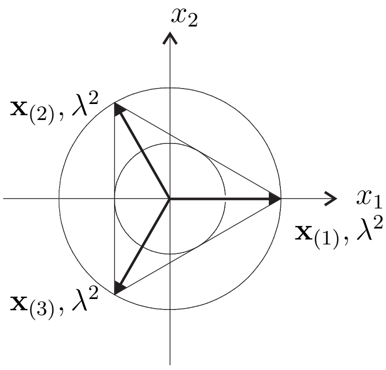

Figure 3. The Poncelet pair of two concentric circles associated with the circularly symmetric $N = 2$ instanton and one member of the porism of triangles with vertices on the outer circle and tangent to the inner circle.

$$\tilde{\psi}_{(1)} = -\frac{2\lambda}{\sqrt{3}}\left[\frac{q_{(1)}}{|q|^3} + \frac{4}{3|q|^4}(-2\hat{\mathbf{x}}\cdot\mathbf{x}_{(2)}q_{(2)} + \hat{\mathbf{x}}\cdot\mathbf{x}_{(1)}q_{(1)} - \hat{\mathbf{x}}\cdot\mathbf{x}_{(1)}q_{(2)} - \hat{\mathbf{x}}\cdot\mathbf{x}_{(2)}q_{(1)})\right]\hat{\bar{q}} + \mathcal{O}\left(\frac{1}{|q|^5}\right) \tag{10.19}$$

$$\tilde{\psi}_{(2)} = -\frac{2\lambda}{\sqrt{3}}\left[\frac{q_{(2)}}{|q|^3} + \frac{4}{3|q|^4}(-2\hat{\mathbf{x}}\cdot\mathbf{x}_{(1)}q_{(1)} + \hat{\mathbf{x}}\cdot\mathbf{x}_{(2)}q_{(2)} - \hat{\mathbf{x}}\cdot\mathbf{x}_{(1)}q_{(2)} - \hat{\mathbf{x}}\cdot\mathbf{x}_{(2)}q_{(1)})\right]\hat{\bar{q}} + \mathcal{O}\left(\frac{1}{|q|^5}\right) \tag{10.20}$$

$$\tilde{\psi}_{(3)} = -\frac{2\lambda}{\sqrt{3}}\left[-\frac{q_{(2)} + q_{(1)}}{|q|^3} + \frac{4}{3|q|^4}(\hat{\mathbf{x}}\cdot\mathbf{x}_{(1)}q_{(1)} + \hat{\mathbf{x}}\cdot\mathbf{x}_{(2)}q_{(2)} + 2(\hat{\mathbf{x}}\cdot\mathbf{x}_{(1)}q_{(2)} + \hat{\mathbf{x}}\cdot\mathbf{x}_{(2)}q_{(1)}))\right]\hat{\bar{q}} + \mathcal{O}\left(\frac{1}{|q|^5}\right). \tag{10.21}$$

These are

$$\psi_{(a)} = \tilde{\psi}_{(1)} + \omega\tilde{\psi}_{(2)} + \omega^2\tilde{\psi}_{(3)} \tag{10.22}$$

$$\psi_{(b)} = \tilde{\psi}_{(1)} + \omega^2\tilde{\psi}_{(2)} + \omega\tilde{\psi}_{(3)} \tag{10.23}$$

with $\omega = \exp(2\pi k/3) = -\frac{1}{2}(1 - \sqrt{3}k)$. To calculate these linear combinations it is useful to note that

$$q_{(1)} = \mathrm{i}\exp(-k\alpha) \qquad q_{(2)} = \omega q_{(1)} \qquad q_{(3)} = \omega^2 q_{(1)}. \tag{10.24}$$

We then find

$$\psi_{(a)} = \frac{4\sqrt{3}\lambda}{|q|^4}\exp(2k\alpha)(-\mathrm{i}\hat{x}_1 + \mathrm{j}\hat{x}_2)\hat{\bar{q}} + \mathcal{O}\left(\frac{1}{|q|^5}\right) \tag{10.25}$$

$$\psi_{(b)} = -\frac{2\sqrt{3}\lambda}{|q|^3}\exp(k\alpha)\mathrm{i}\hat{\bar{q}} + \mathcal{O}\left(\frac{1}{|q|^4}\right). \tag{10.26}$$

$\psi_{(a)}$ is a pure quadrupole and $\psi_{(b)}$ is a pure dipole with no quadrupole. The left-multiplicative factors $\exp(2k\alpha)$ and $\exp(k\alpha)$, respectively, nicely reflect the fact that moving the vertices around the circle corresponds to a gauge transformation. We can now read off the constants $\tilde{b}_\mu$ from $\psi_{(a)}$. Setting $\alpha = 0$ from now on, we have

$$\tilde{\mathbf{b}} = \sqrt{3}\lambda \begin{pmatrix} 0 \\ -\mathrm{i} \\ j \\ 0 \end{pmatrix}. \tag{10.27}$$

Since the gauge-invariant data of this instanton configuration are given by the Poncelet pair of two concentric circles [6] as shown in figure 3, we have circular symmetry in the (x_1, x_2) plane. As in our previous example the solutions $\psi_{(a)}$ and $\psi_{(b)}$, equations (10.25) and (10.26), respectively, exhibit the same symmetry properties as the background. Thus if

$$x_i \mapsto R_{ij}x_j \qquad \text{with} \quad R = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.28}$$

we find that

$$\psi_{(a)} \mapsto \exp(-2k\beta)\psi_{(a)} \tag{10.29}$$

$$\psi_{(b)} \mapsto \exp(-k\beta)\psi_{(b)}. \tag{10.30}$$

Hence, under rotations $\psi_{(a)}$ and $\psi_{(b)}$ are invariant up to a gauge transformation.

Note that the Weyl equation is not invariant under reflections. Yet the background instanton is invariant under reflections in the (x_1, x_2) plane, and also under the reflection $x_3 \mapsto -x_3$. In order to show that the spinor solutions respect this symmetry, we will make use of the fact that two successive reflections correspond to a rotation. We will therefore combine a reflection, say $x_1 \mapsto -x_1$, with the reflection $x_3 \mapsto -x_3$. The combination of these is the 180° rotation $x_i \mapsto R_{ij}x_j$, with

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10.31}$$

We find for the transformation of $\psi_{(a)}$ and $\psi_{(b)}$, respectively,

$$\psi_{(a)} \mapsto \psi_{(a)} \tag{10.32}$$

$$\psi_{(b)} \mapsto -\psi_{(b)}. \tag{10.33}$$

Hence, under this combination of reflections we find $\psi_{(a)}$ and $\psi_{(b)}$ to be invariant up to a gauge transformation. Similarly, there is invariance for any reflection axis in the (x_1, x_2) plane.

So far we have only investigated two highly symmetric $N = 2$ instantons. As the final example, we discuss the case where the circle and ellipse, the gauge-invariant data of an $N = 2$ instanton, are concentric, see figure 4. Note that the moduli space associated with a general $N = 2$ instanton is—excluding rigid gauge transformations—a 13-dimensional manifold whereas the orbits of the conformal group are 12-dimensional. After quotienting out the latter a general $N = 2$ instanton is described by only one free parameter. Indeed it may be proved that *any* $N = 2$ instanton is conformally related to an instanton of which the gauge-invariant data are given by the Poncelet pair of a concentric circle and ellipse, the one

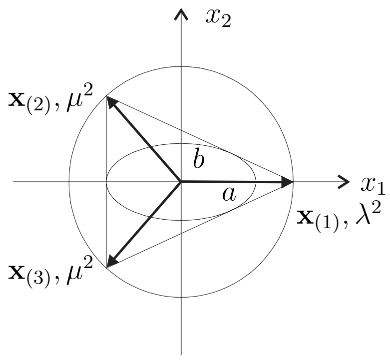

Figure 4. The Poncelet pair of a concentric circle and ellipse associated with the general $N = 2$ instanton and one member of the porism of triangles with vertices on the circle and tangent to the ellipse.

free parameter being the eccentricity a/b of the ellipse. In this sense an $N = 2$ instanton with associated *concentric* circle and ellipse is the most general one.

Suppose, in figure 4, the radius of the circle is $r = 1$. A porism of triangles with vertices on the circle and tangent to the ellipse exists if and only if $a + b = r$. For simplicity we will explicitly choose one member of the porism of triangles, such that $\mathbf{x}_{(1)}$, $\mathbf{x}_{(2)}$ and $\mathbf{x}_{(3)}$ are given by

$$\mathbf{x}_{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{x}_{(2)} = \begin{pmatrix} 0 \\ -a \\ \sqrt{1-a^2} \\ 0 \end{pmatrix} \qquad \mathbf{x}_{(3)} = \begin{pmatrix} 0 \\ -a \\ -\sqrt{1-a^2} \\ 0 \end{pmatrix}. \tag{10.34}$$

The weights associated with the vertices may be calculated using equation (10.6). Scaling the weights such that $\lambda_{(1)}^2 = \lambda^2 = 1$, we find for $\lambda_{(2)}^2$, $\lambda_{(3)}^2$, respectively

$$\lambda_{(2)}^2 = \lambda_{(3)}^2 = \mu^2 = \frac{a}{b}. \tag{10.35}$$

Again we find suitable linear combinations of the three dependent solutions $\tilde{\psi}_{(k)}$, $k = 1, 2, 3$, such that the two resulting solutions form a natural basis of the solution space. The expressions for $\tilde{\psi}_{(k)}$ are similar to equations (10.19), (10.20), (10.21) but a bit more elaborate. The linear combinations are

$$\psi_{(A)} = \tilde{\psi}_{(1)} + k\frac{1}{\sqrt{\Lambda}}(\tilde{\psi}_{(2)} - \tilde{\psi}_{(3)}) \tag{10.36}$$

$$\psi_{(B)} = \tilde{\psi}_{(1)} - k\frac{1}{\sqrt{\Lambda}}(\tilde{\psi}_{(2)} - \tilde{\psi}_{(3)}) \tag{10.37}$$

where $\Lambda = (1-a)/(1+a)$. We find that $\psi_{(A)}$ is again a pure quadrupole, and $\psi_{(B)}$ is a pure dipole. They are, respectively,

$$\psi_{(A)} = \frac{1}{|q|^4}\frac{8a}{\sqrt{\Lambda}}[4(-\mathrm{i}a\hat{x}_1 + \mathrm{j}b\hat{x}_2)\hat{\hat{q}} + (b-a)] + \mathcal{O}\left(\frac{1}{|q|^5}\right) \tag{10.38}$$

$$\psi_{(B)} = -\frac{1}{|q|^3}\frac{8a}{\sqrt{\Lambda}}\mathrm{i}\hat{\hat{q}} + \mathcal{O}\left(\frac{1}{|q|^4}\right). \tag{10.39}$$

It is now easy to read off that the constants $\tilde{b}_\mu$ associated with $\psi_{(A)}$ are

$$\tilde{\mathbf{b}} = \frac{8a}{\Lambda^{1/2}} \begin{pmatrix} 0 \\ -\mathrm{i}a \\ \mathrm{j}b \\ 0 \end{pmatrix}. \tag{10.40}$$

In this example the background field exhibits just a 180° rotational symmetry in the (x_1, x_2) plane and two reflection symmetries, namely under the transformations $x_1 \mapsto -x_1$ and $x_2 \mapsto -x_2$, respectively. Under the 180° rotation in the (x_1, x_2) plane we find for the transformation of $\psi_{(A)}$ and $\psi_{(B)}$, respectively

$$\psi_{(A)} \mapsto \psi_{(A)} \tag{10.41}$$

$$\psi_{(B)} \mapsto -\psi_{(B)}. \tag{10.42}$$

Investigating the symmetry properties of the two solutions under reflections we encounter the same problem as in the case of the circularly symmetric instanton, namely that the Weyl equation itself is not invariant under reflections. However, we may tackle this problem in exactly the same way as before. We will therefore consider the transformation of $\psi_{(A)}$ and $\psi_{(B)}$ under the following two rotations, which are each combinations of two reflections:

$$x_i \mapsto R_{ij} x_j \qquad \text{with} \quad R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{10.43}$$

and

$$x_i \mapsto R_{ij} x_j \qquad \text{with} \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10.44}$$

Thus, the first rotation consists of the combined reflections $x_1 \mapsto -x_1$ and $x_3 \mapsto -x_3$, whereas the second consists of the combined reflections $x_2 \mapsto -x_2$ and $x_3 \mapsto -x_3$. We find for the transformation of $\psi_{(A)}$ and $\psi_{(B)}$ under the rotation given by equation (10.43)

$$\psi_{(A)} \mapsto \psi_{(A)} \tag{10.45}$$

$$\psi_{(B)} \mapsto -\psi_{(B)} \tag{10.46}$$

and under the rotation given by equation (10.44)

$$\psi_{(A)} \mapsto \psi_{(A)} \tag{10.47}$$

$$\psi_{(B)} \mapsto \psi_{(B)}. \tag{10.48}$$

Note that again the two solutions respect the symmetry properties of the background instanton.

11. Conclusion

We have seen that a non-trivial solution to the free Weyl equation in Euclidean 4-space, which is bounded at infinity, necessarily exhibits a singularity at one point at least. In the simplest case such a solution is in quaternionic notation given by

$$\Psi_q = a_q \frac{\bar{q} - \bar{b}_q}{|q - b_q|^4} \tag{11.1}$$

and thus parametrized by two quaternionic constants, a_q and b_q, respectively. Here a_q is the quaternionic analogue of a residue and as such it may be interpreted as the 'spinorial charge' of the spinor wavefunction. The spinor carrying this spinorial charge is clearly localized around b_q, b_q being the position parameter of Ψ_q.

Solutions to the Weyl equation in the background of an arbitrary JNR N-instanton, however, are normalizable and regular in the whole of Euclidean 4-space [4]. Comparing the asymptotic behaviour of these solutions with the asymptotic behaviour of the singular solution to the free Weyl equation, we found that the former solutions can be written up to order $\mathcal{O}(|q|^{-4})$ as linear combinations of the latter. In this sense the introduction of the instanton gauge field results in a delocalization of the spinor. Investigating some special $N = 2$ instantons we were able to show that the parameters describing this delocalized spinor reflect the geometry of the underlying instanton configuration.

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